Linearisable systems and the Gambier approach

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Abstract

A systematic study of the discrete second order projective system is presented, complemented by the integrability analysis of the associated multilinear mapping. Moreover, we show how we can obtain third order integrable equations as the coupling of a Riccati equation with second order Painlevé equations. This is done in both continuous and discrete cases.

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1. Introduction

Integrability is far too general a term. In order to fix the ideas we can just present three most common types of integrability, which suffice in order to explain the properties of the majority of integrable systems [1]. These three types are:

- Reduction to quadrature through the existence of the adequate number of integrals of motion.

- Reduction to linear differential systems through a set of local transformations.

- Integration through IST techniques. This last case is mediated by the existence of a Lax pair (a linear system the compatibility of which is the nonlinear equation under integration) which allows the reduction of the nonlinear equation to a linear integrodifferential one. The above notions can be extended *mutadis mutandis* to the domain of discrete systems.

This paper will focus on the second type of integrability, usually referred to as linearizability. The prototype of the linearizable equations is the Riccati. In differential form this equation writes:

$$w' = \alpha w^2 + \beta w + \gamma \tag{1.1}$$

which is linearisable through a Cole-Hopf transformation. Similarly, the discrete Riccati equation, which assumes the form of a homographic mapping:

$$\overline{x} = \frac{\alpha x + \beta}{\gamma x + \delta} \tag{1.2}$$

where x stands for x_n, $\overline{x}$ for x_{n+1} (and, of course, $\underline{x}$ for x_{n-1}), can be also linearized through a Cole-Hopf transformation.

The extension of the Riccati to higher orders can be and has been obtained [2]. The simplest linearizable system at N dimensions is the projective Riccati which assumes the form:

$$w'_\mu = a_\mu + \sum_\nu b_{\mu\nu} w_\nu + w_\mu \sum_\nu c_\nu w_\nu \quad \text{with } \mu = 1, \ldots, N \tag{1.3}$$

In two dimensions the projective Riccati system can be cast into the second order equation:

$$w'' = -3ww' - w^3 + q(t)(w' + w^2) \tag{1.4}$$

The discrete analog of the projective Riccati does exist and is studied in detail in [3]. The corresponding form is:

$$\overline{x}_\mu = \frac{a_\mu + x_\mu + \sum_\nu b_{\mu\nu} x_\nu}{1 - \sum_\nu c_\nu x_\nu} \tag{1.5}$$

Again in two dimensions the discrete projective Riccati can be written as a second-order mapping of the form

$$\alpha \overline{x} x \underline{x} + \beta \overline{x} x + \gamma x \underline{x} + \delta \overline{x}\underline{x} + \epsilon x + \zeta \overline{x} + \eta \underline{x} + \theta = 0 \tag{1.6}$$

which was first introduced in [4]. The coefficients $\alpha, \beta, \ldots, \theta$ are not totally free. Although the linearizability constraints have been obtained in [4], the study of mappings of the form (1.6) was not complete. In the present work we intend the study of (1.6) in its general form.

Another point must be brought to attention here. In the continuous case the study of second order equations has revealed the relation of the linearizable equations (1.4) to the Gambier equation [5]. The latter is obtained as a system of two coupled Riccati in cascade

$$y' = -y^2 + qy + c \tag{1.7}$$

$$w' = aw^2 + nyw + \sigma$$

where n is an integer. It contains as a special case the linearizable equation (1.4) which is obtained from (1.7) for $n = 1$ and $a = -1$, $c = 0$ and $\sigma = 0$. The discrete analog of the Gambier mapping was introduced in [6] and in full generality in [7].

In the present work we shall also address the question of the construction of integrable third order systems in the spirit of Gambier. Namely we shall start with a second order integrable equation and couple it with a Riccati (or a linear) first order (also integrable) equation. This enterprise may easily assume staggering proportions. In order to limit the scope of our investigation we shall consider coupled systems where the dependent variable enters only in a polynomial way. This leads naturally to the coupling of a Painlevé ($\mathbb{P}$) I or II to a Riccati.

In the next section we shall analyse (1.6) and show how one can isolate the integrable cases through the use of the singularity confinement criterion. In Section 3 we will present how a third order integrable equation can be constructed from the coupling of a Riccati and a Painlevé equation (in the discrete and the continuous case).

2. Linearizable mappings as discrete projective systems

In [4] we have introduced projective system as a way to linearize a second-order mapping. (The general theory of discrete projective systems has been recently presented in [3]). In this older work of ours we have focused on a three-point mapping that can be obtained from a 3×3 projective system. The main idea was to consider the system:

$$\begin{pmatrix} \overline{u} \\ \overline{v} \\ \overline{w} \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \tag{2.1}$$

and conversely

$$\begin{pmatrix} \underline{u} \\ \underline{v} \\ \underline{w} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \tag{2.2}$$

where the matrix M is obviously related to the matrix P through $\overline{M} = P^{-1}$. Introducing the variable $x = u/v$ and the auxiliary $y = w/v$ we can rewrite (2.1) and (2.2) as

$$\overline{x} = \frac{p_{11}x + p_{12} + p_{13}y}{p_{21}x + p_{22} + p_{23}y} \tag{2.3}$$

$$\underline{x} = \frac{m_{11}x + m_{12} + m_{13}y}{m_{21}x + m_{22} + m_{23}y} \tag{2.4}$$

(Since the m_{3i}, p_{3i} do not appear in (2.3), (2.4) and we can simplify M, P by taking $m_{33} = p_{33} = 1$ and $m_{31} = p_{31} = m_{32} = p_{32} = 0$). Finally eliminating y between (2.3) and (2.4) we obtain the mapping:

$$\alpha\overline{x}x\underline{x} + \beta\overline{x}x + \gamma x\underline{x} + \delta\overline{x}\underline{x} + \epsilon x + \zeta\overline{x} + \eta\underline{x} + \theta = 0 \tag{2.5}$$

where the $\alpha, \beta, \ldots \theta$ are related to the m, p's.

Equation (2.5) will be the starting point of the present study. Our question will be when is an equation of this form integrable? (Clearly the relation to the projective system works only for a particular choice of the parameters). In order to investigate the integrability of (2.5) we shall use the singularity confinement approach that was introduced in [8]. What are the singularities of (2.5)? Given the form of (2.5) it is clear that diverging x does not lead to any difficulty. However, another (subtler) difficulty arises whenever x_{n+1} is defined independently of x_{n-1}. In this case the mapping "loses one degree of freedom". Thus the singularity condition is

$$\frac{\partial x_{n+1}}{\partial x_{n-1}} = 0$$

which leads to :

$$(\alpha x + \delta)(\epsilon x + \theta) = (\beta x + \zeta)(\gamma x + \eta) \tag{2.6}$$

Equation (2.6) is the condition for the appearence of a singularity. Given the invariance of (2.5) under homographic transformations it is clear that one can use them in order to simplify (2.6). Several choices exist but the one we shall make here is to choose the roots of (2.6) so as to be equal to 0 and ∞, unless of course (2.6) has two equal roots, in which case we bring them both to 0. Let us examine the distinct root case. For the roots of (2.6) to be 0 and ∞ we must have:

$$\alpha\epsilon = \beta\gamma \tag{2.7}$$

$$\delta\theta = \zeta\eta$$

The generic mapping of the form (2.5) has $\alpha\theta \neq 0$ and we can take $\alpha = \theta = 1$ (by the appropriate scaling of x and a division). We have thus,

$$\overline{x}x\underline{x} + \beta\overline{x}x + \gamma x\underline{x} + \zeta\eta\overline{x}\underline{x} + \beta\gamma x + \zeta\overline{x} + \eta\underline{x} + 1 = 0 \tag{2.8}$$

Nongeneric cases do exist as well and have been examined in detail in [9].

In order to investigate the integrability of the mapping (2.8) we shall apply the singularity confinement criterion. Here the singularities are by construction 0 and ∞. Following the results of [4] we require confinement in just one step. We require to have an indeterminate form 0/0 at the step following the singularity. This leads to the condition $\beta = \zeta = 0$. We thus obtain the mapping:

$$\overline{x}x\underline{x} + \gamma x\underline{x} + \eta\underline{x} + 1 = 0 \tag{2.9}$$

or, solving for $\overline{x}$:

$$\overline{x} = -\gamma + \frac{\eta\underline{x} + 1}{x\underline{x}} \tag{2.10}$$

where γ and η are free. This is indeed integrable: we can show that (2.10) can be obtained from the projective system (2.1), (2.2) provided we take

$$P = \begin{pmatrix} -(\gamma q + 1) & q\overline{q} & 1 \\ q & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.11}$$

and $M = \underline{P}^{-1}$ provided q is some solution of the equation $\overline{q}q\underline{q} + q\underline{q}\eta + \underline{q}\,\underline{\gamma} + 1 = 0$. Integrable but nonlinearizable cases of (2.8) do also exist: they have been identified and presented in detail in [9].

Before closing this section let us present the continuous limits of the linearizable equation we have identified above. For (2.9) we put $x = -1 + \nu w$, $\gamma = 3 + \nu^2 p$, $\eta = \gamma + \nu^3 q$ and we obtain at the limit $\nu \to 0$ the equation:

$$w'' = 3ww' - w^3 + pw + q \tag{2.12}$$

This is equation #6 in the Painlevé/Gambier classification [10] (in noncanonical form) and precisely the one that can be obtained from a $N = 2$ projective Riccati system.

3. Constructing Integrable Third Order Systems: the Gambier Approach

The key idea of Gambier was to construct an integrable second order equation by suitably coupling two integrable first order ones. The latter were well-known: at first order the only integrable (in the sense of having the Painlevé property) ordinary differential equations are either linear or of Riccati type. The Gambier equation is precisely the coupling of two Riccati in cascade.

In [11], we extend this idea for third order systems. We couple Painlevé second order equations with the Riccati equation both in the continuous and the discrete cases.

The coupling of a Painlevé equation with a Riccati was first considered by Chazy. He examined the coupling $\mathrm{P_I}$:

$$w'' = 6w^2 + z,$$

with a Riccati:

$$y' = \alpha y^2 + \lambda w + \gamma \tag{3.1}$$

where $\alpha, \beta, \lambda, \gamma$ are functions of z. This coupling is additive: it is indeed the only coupling that is compatible with integrability. Chazy found that (3.1) must have the form:

$$y' = \frac{1-k^2}{4} y^2 + w + \gamma, \tag{3.2}$$

where $k = 6m + n$. Chazy found the following necessary integrability constraints:

$$\begin{aligned} n &= 2 \quad \gamma = 0 \\ n &= 3 \quad \gamma' = 0 \\ n &= 4 \quad \gamma'' = \mu\gamma^2 + \nu z \\ n &= 5 \quad \gamma''' = \mu\gamma\gamma' + \nu, \end{aligned}$$

where μ and ν are specific numerical constants. It turns out that for $k = n$ they are also sufficient. For $k = 6m + 1$ the first condition appears at $k = 7$. In this case the constraint reads:

$$\gamma^{(5)} = 48\gamma\gamma''' + 120\gamma'\gamma'' - \frac{2304}{5}\gamma'\gamma^2 - 24z\gamma' - 48\gamma.$$

This equation has the Painlevé property and is thus expected to be integrable.

In [11] we have presented the coupling of a Riccati to other integrable second order differential equations.

To construct integrable discrete systems in the same spirit as Gambier we need a detailed knowledge of the forms of the equations to be coupled and an integrability detector. The second order mappings which play the role of the Painlevé equations in the discrete domain have been the object of numerous detailed studies and we are now in possesion of discrete forms of all the equations of the Painlevé/Gambier classification. The discrete integrability detector is based on the singularity confinement.

We consider the coupling of a discrete Riccati for the variable y:

$$\overline{y} = \frac{(\alpha x + \beta)y + (\eta x + \theta)}{(\epsilon x + \zeta)y + (\gamma x + \delta)}, \tag{3.3}$$

(where α, β, ...,θ depend in general on n) the coefficients of which depend linearly on x, the solution of the discrete $\mathrm{P_{II}}$. The mapping (3.3) can be brought under canonical form through the application of homographic transformations on y to either:

$$\overline{y} = \frac{(\alpha x + \beta)y + 1}{y + (\gamma x + \delta)}. \tag{3.4}$$

or:

$$\overline{y}(\gamma x + \delta) - y(\alpha x + \beta) - 1 = 0. \tag{3.5}$$

In [11] we examined in detail the coupling of (3.4) and (3.5) with either d-$\mathrm{P_I}$ or d-$\mathrm{P_{II}}$ (under various forms). Here, we will focus on a particular example of a coupling to d-$\mathrm{P_{II}}$.

How does one apply the singularity confinement criterion to a mapping such as (3.4) when x is given by some discrete equation like d-$\mathrm{P_I}$ or d-$\mathrm{P_{II}}$? The singularity manifests itself by the fact that $\overline{y}$ is independent of y i.e. when

$$(\gamma x + \delta)(\alpha x + \beta) = 1. \tag{3.6}$$

This quadratic equation has two roots X_1, X_2. The confinement condition is for y to recover the lost degree of freedom. This can be done if y assumes an indeterminate form $0/0$. This means that x at this stage must again satisfy (3.6) and moreover be such that the denominator (or, equivalently, the numerator) vanishes.

Let us assume now that for some n we have $x_n = X_1$. The confinement requirement is that k steps later $x_{n+k} = X_2$. Starting from $x_n = X_1$ and some initial datum x_{n-1}, we can iterate the mapping for x and obtain x_{n+k} as a complicated function of x_{n-1} and X_1. Since x_{n+k} depends on the free parameter x_{n-1} there is no hope for x_{n+k} to be equal to X_2 if X_1 is a generic point for the mapping of x. The only possibility is that both X_1 and X_2 be special values. What are the special values of this equation depends

on its details, but clearly in the case of the discrete Painlevé's we shall examine here, these values can only be the ones related to the singularities. To be more specific, let us examine d-$\mathrm{P_{II}}$:

$$\overline{x} + \underline{x} = \frac{zx + a}{1 - x^2}. \tag{3.7}$$

The only special values of x are the ones related to the singularity $x_n = \pm 1$, $x_{n+1} = \infty$, $x_{n+2} = \mp 1$ while $\ldots$, x_{n-2}, x_{n-1} and x_{n+3}, x_{n+4}, $\ldots$ are finite. This means that the two roots of (3.6) must be two of $\{+1, \infty, -1\}$ and moreover that confinement must occur in two steps. The precise implementation of singularity confinement requires that the denominator of (3.4) at $n+2$ vanishes (and because of (3.6) this ensures that the numerator vanishes as well). Moreover, we must make sure that the lost degree of freedom (i.e. the dependence on y) is indeed recovered through the indeterminate form.

The singularity patterns of (3.7) are

$$\{\pm 1, \infty, \mp 1\}. \tag{3.8}$$

This means that the singularity condition (3.4) must have ± 1 as roots. As a result we have:

$$\delta = -\beta/(\alpha^2 - \beta^2), \tag{3.9}$$

$$\gamma = \alpha/(\alpha^2 - \beta^2). \tag{3.10}$$

The two different patterns lead to a first confinement conditions given by:

$$\beta = k\alpha$$

where k is a constant with binary freedom which we will ignore from now. The second condition:

$$\underline{\alpha}\alpha^2\overline{\alpha} = \frac{1}{(1 - k^2)^2}. \tag{3.11}$$

This equation can be solved by linearisation just by taking the logarithm of both sides. More examples of couplings of discrete equations can be found in [11].

3. Conclusion

In the previous sections we have first investigated 3-point mappings that are integrable through linearization. Our analysis was guided by the analogy with the continuous situation and results of ours on $N = 2$ projective systems. We have presented an analysis of the linearizable mapping and identified one of its integrable form. Moreover we have presented an approach for the construction of integrable third order systems through the coupling of a second order equation to a Riccati or a linear first order equation. Thus we have extended the Gambier approach (first used in his derivation of the second order ODE that bears his name) to higher order systems. We have applied this coupling method to both continuous and discrete systems.

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